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Cubic Splines on the Real Line[1]

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We shall limit ourselves to the consideration of cubic splines since we wish to take full advantage of the special properties of the system of linear equations which arises in this situation. For the same reason we restrict our considerations to *simple splines* of interpolation. These splines are required by definition to interpolate only to $f(x)$ at mesh points and not to both $f(x)$ and $f'(x)$; they possess continuous second derivatives throughout their domain of definition.

For most purposes, the restriction to simple cubic splines is innocuous since at a mesh point where interpolation to both $f(x)$ and $f'(x)$ is required, a cubic spline of interpolation is separated into two independent splines. For instance, if there are two double points of interpolation x_{j_0} and x_{j_1} such that $x_{j_0} < x_{j_1}$, the spline reduces on the closed interval $[x_{j_0}, x_{j_1}]$ to a type I spline of interpolation ([*1*], p. 75) to $f(x)$; the theory of such splines is well known. If, however, there is no point of double interpolation x_{j_1} to the right of x_{j_0}, we must consider splines $S_\Delta(f;x)$ on the infinite interval $[x_{j_0}, \infty)$ which interpolate to $f(x)$ at the mesh points of a mesh Δ on (x_{j_0}, ∞). We shall consider this situation in detail, and also indicate the rather minor modifications required when the interval (x_{j_0}, ∞) is replaced by the interval $(-\infty, \infty)$.

In the interval $x_{i-1} \leqslant x \leqslant x_i$ a cubic spline $S_\Delta(f;x)$ which interpolates to the values $f_j = f(x_j)$, $j = 0, 1, \ldots$, at the points x_j of the mesh $\Delta : x_0 < x_1 < \ldots$ is given ([*1*], p. 10) by

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$$S_\Delta(f;x) = [M_{i-1}(x_i - x)^3 + M_i(x - x_{i-1})^3]\frac{1}{6l_i} + \left[\frac{f_i}{l_i} - M_i\frac{l_i}{6}\right](x - x_{i-1}) + \left[\frac{f_{i-1}}{l_i} - \frac{M_{i-1}l_i}{6}\right](x_i - x). \tag{1}$$

Here $l_j = x_j - x_{j-1}$ and $M_j = S''_\Delta(f;x_j)$. As a consequence of (1) we have on $x_{i-1} \leqslant x \leqslant x_i$

$$S'_\Delta(f;x) = -\frac{M_{i-1}(x_i - x)^2}{2l_i} + \frac{M_i(x - x_{i-1})^2}{2l_i} + \frac{f_i - f_{i-1}}{l_i} - \frac{M_i - M_{i-1}}{6} l_i, \tag{2}$$

$$S''_\Delta(f;x) = \frac{M_i(x - x_{i-1})}{l_i} + \frac{M_{i-1}(x_i - x)}{l_i}. \tag{3}$$

It is immediate from (1) that $S_\Delta(f;x_i) = f_i$ and from (3) that $S''_\Delta(f;x_i) = M_i$ $(i = 0,1,\ldots)$. Consequently, $S_\Delta(f;x)$ and $S''_\Delta(f;x)$ are in $C(x_0,\infty)$. In order to insure that $S'_\Delta(f;x)$ is also in $C(x_0,\infty)$, the following infinite system of equations must be satisfied:

$$\tfrac{1}{6}l_i M_{i-1} + \tfrac{1}{3}(l_i + l_{i+1}) M_i + \tfrac{1}{6}l_{i+1} M_{i+1} = \frac{f_{i+1} - f_i}{l_{i+1}} - \frac{f_i - f_{i-1}}{l_i} \equiv d_i \qquad (i = 1,2,\ldots). \tag{4}$$

In order that $S'_\Delta(f;x_0) = f_0' \equiv f'(x_0)$ we have to satisfy the additional equation

$$\frac{1}{3}l_1 M_0 + \frac{1}{6}l_1 M_1 = \frac{f_1 - f_0}{l_1} - f_0' \equiv d_0. \tag{5}$$

In matrix form we must solve the equation

$$AM = d, \tag{6}$$

where

$$A = \begin{pmatrix} \frac{1}{3}l_1 & \frac{1}{6}l_1 & 0 & 0 & \cdot & \cdot & \cdot \\ \frac{1}{6}l_1 & \frac{1}{3}(l_1 + l_2) & \frac{1}{6}l_2 & 0 & \cdot & \cdot & \cdot \\ 0 & \frac{1}{6}l_2 & \frac{1}{3}(l_2 + l_3) & \frac{1}{6}l_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \tag{7}$$

$$M = (M_0, M_1, \ldots)^T, \tag{8}$$

$$d = (d_0, d_1, \ldots)^T. \tag{9}$$

When the interval (x_0, ∞) is replaced by the real line $(-\infty, \infty)$, Eqs. (4) apply except that now $i = 0, \pm 1, \pm 2, \ldots$, and Eq. (5) is omitted. In this case

$$A = \begin{pmatrix} \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \frac{1}{6}l_{-1} & \frac{1}{3}(l_{-1}+l_0) & \frac{1}{6}l_0 & 0 & 0 & \cdot \\ \cdot & \cdot & 0 & \frac{1}{6}l_0 & \frac{1}{3}(l_0+l_1) & \frac{1}{6}l_1 & 0 & \cdot \\ \cdot & \cdot & 0 & 0 & \frac{1}{6}l_1 & \frac{1}{3}(l_1+l_2) & \frac{1}{6}l_2 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \tag{7'}$$

$$M = (\ldots, M_{-1}, M_0, M_1, \ldots)^T, \tag{8'}$$

$$d = (\ldots, d_{-1}, d_0, d_1, \ldots)^T. \tag{9'}$$

Moreover, if we reorder the components of M (and d) such that in rearranged form

$$M = (M_0, M_{-1}, M_1, M_{-2}, M_2, \ldots),$$

then A will be a singly-infinite matrix such as (6) which is symmetric and has the same diagonal elements as those of A, although they will be reordered.

Let us introduce the notation

$$\begin{aligned} \|\Delta\| &= \sup_i l_i, \\ \delta_\Delta &= \inf_i l_i, \\ \mathscr{R}_\Delta &= \|\Delta\|/\delta_\Delta,\ \delta_\Delta \neq 0. \end{aligned} \tag{10}$$

If we require $\delta_\Delta > 0$, then $A = (a_{ij})$, as given by (7) or (7′), is a symmetric matrix, which is strongly diagonally dominant in the sense that

$$\inf_i \left\{ |a_{ii}| - \sum_{j \neq 1} |a_{ij}| \right\} \geqslant \frac{1}{3}\delta_\Delta > 0. \tag{11}$$

Moreover, $S_\Delta(f; x)$ exists and is unique if and only if Eq. (6) has a unique solution. In addition, the standard convergence results ([1], Chapter II) can be obtained for the intervals $[x_0, \infty)$ and $(-\infty, \infty)$ if

$$\|A^{-1}\|_\infty = \sup_j \sum_j |b_{ij}| < K/\delta_\Delta, \tag{12}$$

where K is a constant independent of Δ and

$$A^{-1} = (b_{ij}). \tag{13}$$

By l^∞ we shall mean the collection of all vectors

$$v = (v_0, v_1, \ldots)^T, \tag{14}$$

where v_j $(j = 0, 1, \ldots)$ is a complex number and $\|v\|_\infty \equiv \sup_j |v_j| < \infty$. The set l^∞ is a Banach space under this norm. If $A = (a_{ij})$, $(i, j = 0, 1, 2, \ldots)$ is a matrix for which

$$\sup_i \sum_{j=0}^{\infty} |a_{ij}| < \infty,$$

then A defines a bounded linear transformation on l^∞; and if $\|A\|_\infty$ is the infimum of all positive numbers C for which

$$\|Av\|_\infty \leqslant C\|v\|_\infty, \tag{15}$$

then

$$\|A\|_\infty = \sup_i \sum_{j=0}^{\infty} |a_{ij}|. \tag{16}$$

We shall denote by l^2 the set of all vectors (14) for which

$$\|v\|^2 = \sum_{j=0}^{\infty} |v_j|^2 < \infty.$$

The space l^2 is a Hilbert space under the inner product

$$(v, u) = \sum_{j=0}^{\infty} v_j \bar{u}_j.$$

A matrix $A = (a_{ij})$ defines a linear transformation of l^2 into l^2 if Av is in l^2 whenever v is in l^2. We shall let $\|A\|$ denote the infimum of all positive numbers C for which

$$\|Av\| \leqslant C\|v\|. \tag{17}$$

If $\|A\|$ is finite, A defines a bounded linear transformation of l^2 into l^2. Although we lack a convenient expression for $\|A\|$, Schur's theorem ([*2*], p. 328) asserts

$$\|A\| \leqslant \left\{\left(\sup_i \sum_{j=0}^{\infty} |a_{ij}|\right)\left(\sup_j \sum_{i=0}^{\infty} |a_{ij}|\right)\right\}^{1/2}. \tag{18}$$

If A is either symmetric or Hermitian, (18) becomes

$$\|A\| \leqslant \|A\|_\infty. \tag{19}$$

The following theorem establishes the existence of $S_\Delta(f; x)$ if $\|\Delta\| < \infty$, $\delta_\Delta > 0$, and shows that (12) is satisfied so that convergence results can be derived.

THEOREM. *Let* $A = (a_{ij})$ $(i, j = 0, 1, \ldots)$ *be a real symmetric matrix for which*

$$\begin{gathered}\inf_i \left\{|a_{ii}| - \sum_{j \neq i} |a_{ij}|\right\} \equiv \delta > 0,\\ \sup_i \sum_{j=0}^{\infty} |a_{ij}| \equiv \|A\|_\infty < \infty.\end{gathered} \tag{20}$$

Under these conditions, A^{-1} exists and, with the notation $A^{-1}=(b_{ij})$, we have

$$\|A^{-1}\|_\infty \equiv \sup_i \sum_{j=0}^{\infty} |b_{ij}| \leqslant \frac{1}{\delta}. \tag{21}$$

Proof. Consider, momentarily, A as a linear transformation on l^2. Since (19) holds, A is a bounded linear transformation, and since A is Hermitian, its spectrum is real; thus for $\lambda > 0$, $A_\lambda^{-1} \equiv (A - i\lambda I)^{-1}$ exists. Moreover, for positive λ we have

$$\inf_i \left\{ |a_{ii} - i\lambda| - \sum_{j \neq i} |a_{ij}| \right\} \equiv \delta_\lambda > 0. \tag{22}$$

In addition $\delta_\lambda \to \delta$ as $\lambda \to 0$. Now, A_λ^{-1} is a bounded linear operator on l^2 and can be represented by an infinite matrix $(a_{ij}^{(-\lambda)})$ which is symmetric since A_λ, while not Hermitian, is symmetric. Let v be in l^2 and choose $\epsilon > 0$. There exists i_ϵ such that $\|v\|_\infty - |v_{i_\epsilon}| < \epsilon$. Consequently, if $A_\lambda = (a_{ij}^\lambda)$,

$$\begin{aligned} \|A_\lambda v\|_\infty &\geqslant |a_{i_\epsilon i_\epsilon}^\lambda|\,|v_{i_\epsilon}| - \|v\|_\infty \sum_{j \neq i_\epsilon} |a_{i_\epsilon j}^\lambda| \\ &\geqslant \|v\|_\infty \left\{ |a_{i_\epsilon i_\epsilon}^\lambda| - \sum_{j \neq i_\epsilon} |a_{i_\epsilon j}^\lambda| \right\} - |a_{i_\epsilon i_\epsilon}^\lambda|\,\epsilon. \end{aligned} \tag{23}$$

Since (23) holds for all $\epsilon > 0$ and since $|a_{i_\epsilon i_\epsilon}^\lambda| < \|A_\lambda\|_\infty$, we have

$$\|A_\lambda v\|_\infty \geqslant \delta_\lambda \|v\|_\infty. \tag{24}$$

Indeed the inequality (24) holds not only for v in l^2 but for any v in l^∞. However, for ω in l^2 we can choose v such that $v = A_\lambda^{-1}\omega$, and obtain

$$\|A_\lambda^{-1}\omega\|_\infty \leqslant \frac{1}{\delta_\lambda}\|\omega\|_\infty. \tag{25}$$

Now let

$$Y_k^N = (\exp(-i \text{ any } a_{k,0}^{(-\lambda)}), \exp(-i \text{ any } a_{k,1}^{(-\lambda)}), \ldots, \exp(-i \text{ any } a_{k,n}^{(-\lambda)}), 0, 0, \ldots)^T. \tag{26}$$

Since Y_k^N is in l^2 and $\|Y_k^N\|_\infty = 1$, it follows that

$$|a_{k,0}^{(-\lambda)}| + |a_{k,1}^{(-\lambda)}| + \ldots + |a_{k,N}^{(-\lambda)}| \leqslant \frac{1}{\delta_\lambda}. \tag{27}$$

But this is true for $N = 0, 1, \ldots$, and for each k. Hence,

$$\sup_k \sum_{j=0}^{\infty} |a_{kj}^{(-\lambda)}| \leqq \frac{1}{\delta_\lambda} < \infty,$$

and A_λ^{-1} is actually a bounded linear transformation on l^∞ which is inverse to A_λ. Moreover, we can choose λ such that

$$\|A - A_\lambda\|_\infty = |\lambda| < \delta_\lambda \leqslant \frac{1}{\|A_\lambda^{-1}\|_\infty};$$

hence ([2], p. 164) A^{-1} exists. Furthermore, we have for every $\lambda > 0$

$$\|A^{-1}\|_\infty \leqslant \frac{\|A_\lambda^{-1}\|_\infty}{1 - \|A_\lambda^{-1}\|_\infty \lambda}.$$

If we let $\lambda \to 0$, then

$$\|A^{-1}\|_\infty \leqslant \frac{1}{\delta},$$

which proves the theorem.

A simpler proof can be given in the case $\mathscr{R}_\Delta < 2$. Let D be the diagonal matrix whose diagonal is that of A. Then D^{-1} exists and

$$\|D^{-1}\|_\infty < \frac{3}{2\delta_\Delta}.$$

In addition, we have

$$\|A - D\|_\infty < \frac{\|\Delta\|}{3} = \frac{\mathscr{R}_\Delta}{3}\delta_\Delta < \frac{2\delta_\Delta}{3} < \frac{1}{\|D^{-1}\|_\infty}$$

under the condition $\mathscr{R}_\Delta < 2$. The theorem now follows ([2], p. 164).

We have demonstrated that if a sequence of $\{y_i\}$ $(i = 0, 1, \ldots)$ is prescribed such that for the associated vector d we have $\|d\|_\infty < \infty$, then there is a unique cubic spline $S_\Delta(x)$ interpolating to y_i at x_i, having a prescribed value y_0' for $S'_\Delta(x)$ at x_0, and having $\|M\|_\infty < \infty$. If we do not require $\|d\|_\infty < \infty$, then there is a one-parameter family of splines on $[x_0, \infty]$ having these properties. This s also true even if $\|d\|_\infty < \infty$; but in this case, $\|M\|_\infty = \infty$ except for one value of the parameter. In support of these assertions we observe that there is a unique cubic $C(x)$ on the interval $[x_0, x_1]$ with $C(x_0)$, $C'(x_0)$, $C''(x_0)$, $C(x_1)$ arbitrarily prescribed. Using the values of $C'(x)$ and $C''(x)$ at x_1 we now (by repeating the construction) can extend the domain of definition of $C(x)$ to $[x_0, x_2]$ and have $C(x)$ in $C^2[x_0, x_2]$. Further repetition of this construction gives the desired spline function. Similarly, $C(x)$ can be extended to the interval $(-\infty, \infty)$. In this case there are two more degrees of freedom $(C'(x_0), C''(x_0))$ than when we require $\|d\|_\infty < \infty$, $\|M\|_\infty < \infty$. When $\|d\|_\infty < \infty$ in order to avoid a contradiction to our earlier existence theorem, $\|M\|_\infty = \infty$ with one exception.

The following inequalities typify the behavior of $C(x)$: If $C''(x) \leqslant Q$ for $x > x_0 = 0$, then $C(x) \leqslant y_0 + y_0'x + \frac{1}{2}Qx^2$ and $C'(x) \leqslant y_0' + Qx$ for $x > 0$. Similarly, if $C''(x) \geqslant Q_1$ for $x > x_0 = 0$, then $C(x) \geqslant y_0 + y_0'x + \frac{1}{2}Q_1x^2$ and $C'(x) \geqslant y_0' + Q_1x$ for $x > 0$. These inequalities are the best possible since they become equalities if $C(x)$ is a quadratic. Of course $C''(x_i) \leqq Q$ for $i \geqq 0$ implies $C''(x) \leqq Q$ for $x \geqq x_0$; $C''(x_i) \geqq Q_1$ for $i \geqq 0$ implies $C''(x) \geqq Q_1$ for $x \geqq x_0$.

References

1. J. H. Ahlberg, E. N. Nilson, and J. L. Walsh, "The Theory of Splines and Their Applications". Academic Press, New York, 1967.

2. A. E. Taylor, "Introduction to Functional Analysis". Wiley, New York, 1958.